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Duality in graph families

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Abstract

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The concepts of complete and free families are studied by Catlin. It has been noted by Catlin that these concepts are, in some sense, dual to each other. In this paper, we investigate the duality between edge-contractions and edge-deletions, and establish some results dual to several Catlin's theorems.

Introduction

We shall follow the notation of Bondy and Murty [1], except that we shall not allow graphs to have loops. For a graph G with a connected subgraph H, the *contraction* G/H is the graph obtained from $G - V(H)$ by adding a vertex v_H and by adding a set of edges that collectively join each $w \in V(G) - V(H)$ to v_H as many times as w and $V(H)$ are joined by edges in G. In general, if $H_1, H_2, \ldots, H_c$ are connected components of H with $E(H) = \bigcup_{i=1}^{c} E(H_i)$ and if H is a subgraph of G, then G/H denotes $(\cdots((G/H_1)/H_2)/\cdots)/H_c$. We shall use $\mathbb{N}$ to denote the set of positive integers.

A collection $\mathcal{S}$ of graphs is called a *graph family* or a *family*. If H is a subgraph of G, we denote this by $H \subseteq G$. Call a family $\mathcal{S}$ of graphs *closed under contraction* if

$$H \subseteq G,\ H \text{ connected},\ G \in \mathcal{S} \Rightarrow G/H \in \mathcal{S}.$$

Call a family $\mathcal{C}$ of complete graphs *complete* if $\mathcal{C}$ satisfies these three axioms:

(C1) $K_1 \in \mathcal{C}$;
(C2) $\mathcal{C}$ is closed under contraction;
(C3) $H \subseteq G,\ H \in \mathcal{C},\ G/H \in \mathcal{C} \Rightarrow G \in \mathcal{C}$.

A nontrivial example of a typical complete family of connected graphs is

$$\mathscr{C} = \{\text{2-edge-connected graphs}\} \cup \{K_1\}.$$

Call a family $\mathscr{F}$ of graphs *free* if these three axioms hold:

(F1) $K_1 \in \mathscr{F}$;

(F2) $H \subseteq G,\ G \in \mathscr{F} \Rightarrow H \in \mathscr{F}$;

(F3) For any induced connected subgraph H of G, $H \in \mathscr{F}$ and $G/H \in \mathscr{F}$ imply $G \in \mathscr{F}$.

As an example of a typical free family of graphs, we have the following:

$$\mathscr{F} = \{\text{forests}\}.$$

A family $\mathscr{C}$ of connected graphs is called *near-complete* if $\mathscr{C}$ satisfies (C1), (C3), and

(C2′) $H \subseteq G,\ H \in \mathscr{C},\ G \in \mathscr{C} \Rightarrow G/H \in \mathscr{C}$.

An example of graph family which is near-complete but not complete is

$$\{G\colon G \text{ is connected with odd order}\}.$$

A family $\mathscr{S}$ is called *closed under edge-addition* if for any graph G and edge $e \in E(G)$, $G - e \in \mathscr{S}$ implies $G \in \mathscr{S}$. For any family $\mathscr{S}$ of graphs, we define the *kernel* of $\mathscr{S}$ to be the family of connected graphs

$$\mathscr{S}^{o} = \{H\colon \text{For every supergraph } G \text{ of } H,\ G \in \mathscr{S} \Leftrightarrow G/H \in \mathscr{S}\}.$$

Trivially, $K_1 \in \mathscr{S}^{o}$. If $\mathscr{S}^{o} = \{K_1\}$, then we call $\mathscr{S}^{o}$ *trivial.*

Example 1 (Kernel of a family of graphs). For a family $\mathscr{S}$ of graphs, the concept of the kernel $\mathscr{S}^{o}$ is important because it can be (and has been) used in a reduction method to determine membership in $\mathscr{S}$: if $H \subseteq G$ and $H \in \mathscr{S}^{o}$ then

$$G \in \mathscr{S} \Leftrightarrow G/H \in \mathscr{S}. \tag{2}$$

As indicated in [5, 6], (2) can be used for the case when $\mathscr{S}$ is the family of graphs having a spanning closed trail (*supereulerian graphs, see* [6]). *If H is collapsible* (as first defined by Catlin in [6]), then H is in the kernel of $\mathscr{S}$. Hence if H is a collapsible subgraph of G then (2) holds. The reduction method is especially helpful for supereulerian graphs, because no characterization of them is known. This reduction has been applied by Catlin and H.-J. Lai to study supereulerian graphs and hamiltonian line graphs in a series of papers ([4, 8–11, 13–17], among others).

An *elementary homomorphism* of a graph G is a graph G' obtained from G by identifying two vertices lying in the same component in G, and by deleting any loops that might result. (Note that this is not the usual definition.) A *homomorphism* of G is a graph obtained from G by a sequence of elementary homomorphisms. If H is a homomorphism image of G, then we say that G is *homomorphic* to H.

Let $\mathcal{S}$ be a graph family. We say that $\mathcal{S}$ is *closed under homomorphisms* if $G \in \mathcal{S}$ implies that every homomorphism of G is in $\mathcal{S}$. Also we call $\mathcal{S}$ *closed under deletion* if $G \in \mathcal{S}$ and $H \subseteq G$ together imply $H \in \mathcal{S}$. Denote by $\mathcal{S}^{\mathrm{R}}$ the family of all graphs having no nontrivial connected subgraph in $\mathcal{S}$. (If $\mathcal{S}$ contains no connected graph then define $\mathcal{S}^{\mathrm{R}} = \{\text{all graphs}\}$.) Denote by $\mathcal{S}^{\mathrm{C}}$ the family of all connected graphs that cannot be contracted onto a nontrivial member of $\mathcal{S}$. Denote by $\mathcal{S}^{\mathrm{H}}$ the family of connected graphs having no homomorphism onto a nontrivial member of $\mathcal{S}$.

In [2] and [3] (see also [5, 7]), Catlin had shown the following.

Theorem A. *Let $\mathcal{S}$ be a graph family. Then:*
(a) *$K_1 \in \mathcal{S}$ if and only if $\mathcal{S}^{\mathrm{o}} \in \mathcal{S}$;*
(b) $(\mathcal{S}^{\mathrm{o}})^{\mathrm{o}} = \mathcal{S}^{\mathrm{o}}$.

Theorem B. *If $\mathcal{C}$ is a near-complete family of graphs, then:*
(a) $\mathcal{C} = \mathcal{C}^{\mathrm{o}}$;
(b) *$\mathcal{C}$ is closed under edge-addition (i.e., $G - e \in \mathcal{C} \Rightarrow G \in \mathcal{C}$).*

Theorem C. *For any graph family $\mathcal{S}$ if $\mathcal{S}$ or $\mathcal{S}^{\mathrm{o}}$ is closed under contraction, then $\mathcal{S}^{\mathrm{o}}$ is complete.*

Theorem D. *Let $\mathcal{C}$ be a family of graphs closed under contraction. These are equivalent;*
(a) *$\mathcal{C}$ is the kernel of some graph family closed under contraction:*
(b) *$\mathcal{C}$ is a complete family;*
(c) $\mathcal{C} = \mathcal{C}^{\mathrm{o}}$.

We say that a graph G is *t edges short of being in $\mathcal{S}$* if $t \in \mathbb{N} \cup \{0\}$ is the least number, such that some set of t edges can be added to G to obtain a spanning supergraph G', where $G' \in \mathcal{S}$.

Let $\mathcal{C}$ be a complete family. Define

$$\mathcal{C}(t) = \{G\colon G \text{ is at most } t \text{ edges short of being in } \mathcal{C}\}.$$

Catlin in [2, 3] presents many useful examples. Among them are the following.

Example 2 (spanning trees and strength)**.** The *strength* of a graph G is defined by (see [12])

$$\eta(G) = \min_{E \subseteq E(G)} \frac{|E|}{\omega(G - E) - 1}, \tag{3}$$

where $G \neq K_1$ and the minimum in (3) runs over all subsets $E \subseteq E(G)$ such that $\omega(G - E)$, the number of components of $G - E$, is at least 2. Note that for any graph G, $\lfloor \eta(G) \rfloor$ is the minimum number of edge-disjoint spanning trees of G (see [12, 18, 20]). Define

$$\mathcal{S}_{r,0} = \{G\colon \eta(G) \geqslant r\} \cup \{K_1\},$$

for any rational number $r \geqslant 1$; and if also $t \in \mathbb{N} \cup \{0\}$, then define

$$\mathcal{S}_{r,0}(t) = \{G\colon G \text{ is at most } t \text{ edges short of being in } \mathcal{S}_{r,0}\}.$$

When $r \in \mathbb{N}$, $\mathcal{S}_{r,0}$ is the family of graphs containing r edge-disjoint spanning trees.

Theorem E. *Let r be a rational number at least* 1. *If $t \in \mathbb{N} \cup \{0\}$, then the kernel of $\mathcal{S}_{r,0}(t)$ is*

$$\mathcal{S}^{\mathrm{o}}_{r,0}(t) = \mathcal{S}_{r,0} = \{G\colon \eta(G) \geqslant r\} \cup \{K_1\}.$$

Example 3 (more strength). Define, for any rational $r \geqslant 1$,

$$\Gamma_{r,0} = \{G : \eta(G) > r\} \cup \{K_1\},$$

and also for any $t \in \mathbb{N} \cup \{0\}$, define

$$\Gamma_{r,0}(t) = \{G\colon G \text{ is at most } t \text{ edges short of being in } \Gamma_{r,0}\}.$$

Define the *fractional arboricity* of G by (see [12])

$$\gamma(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H) - 1|},$$

where $G \neq K_1$ and the maximum runs over all subgraphs H of G with $|V(H)| \geqslant 2$. We call $\lceil \gamma(G) \rceil$ the *arboricity* of G. Nash-Williams [19] showed that $\lceil \gamma(G) \rceil$ is the minimum number of spanning trees whose union contains G. When $r \in \mathbb{N}$, $\Gamma^{R}_{r,0}$ is the family of graphs with arboricity at most r.

Theorem F. *If $r \geqslant 1$ is rational and $t \in \mathbb{N} \cup \{0\}$, then $\Gamma_{r,0}(t)$ has the kernel*

$$\Gamma^{\mathrm{o}}_{r,0}(t) = \Gamma_{r,0} = \{G\colon \eta(G) > r\} \cup \{K_1\}.$$

Example 4 (edge-connectivity). For any $r \in \mathbb{N} \cup \{0\}$, define

$$\mathcal{S}(r, 0) = \{G : \kappa'(G) \geqslant r\} \cup \{K_1\},$$

and define, for any $r, t \in \mathbb{N} \cup \{0\}$,

$$\mathcal{S}(r, 0)(t) = \{G\colon G \text{ is at most } t \text{ edges short of being in } \mathcal{S}(r, 0)\}.$$

It is routine to check that $\mathcal{S}(r, 0)$ is a complete family of graphs. It is characterized by a well-known variant of Menger's Theorem (see [1, p. 204]).

Theorem G. $\mathcal{S}^{\mathrm{o}}(r, 0)(t) = \mathcal{S}(r, 0)$.

Also, Catlin ([2, 3, 5, 7]) had proved the following.

Theorem H. *If $\mathscr{C}$ is a complete family, then either $\mathscr{C} = \{K_1\}$ or $\mathscr{C}$ contains some spanning supergraphs of complete graphs K_n of every order $n \in \mathbb{N}$.*

Theorem I. *A complete family is closed under homomorphisms.*

Theorem J. *Let $\mathscr{S}$ be a family of graphs.*
(a) *If $\mathscr{S}$ is closed under deletion, then $\mathscr{S}^{\mathrm{H}}$ is complete.*
(b) *If $\mathscr{S}$ is closed under homomorphisms, then $\mathscr{S}^{\mathrm{R}}$ is free.*

Theorem K. *Let $\mathscr{S}$ be a family of graphs.*
(a) *If $\mathscr{S}$ is complete, then $\mathscr{S}^{\mathrm{R}}$ is a free family and $\mathscr{S} = (\mathscr{S}^{\mathrm{R}})^{\mathrm{C}}$.*
(b) *If $\mathscr{S}$ is free, then $\mathscr{S}^{\mathrm{C}}$ is a complete family, and if $\mathscr{S}$ is not a finite family of edgeless graphs, then $\mathscr{S} = (\mathscr{S}^{\mathrm{C}})^{\mathrm{R}}$.*

It is an obvious phenomenon that there is some kind of duality among graph families. This duality takes the form that the contraction and deletion are dual concepts of each other. This differs from the matroid dual since it is a duality among graphs, whereas the matroid dual of a graphic matroid may not be graphic (see [21]). In this paper, we shall further study the complete families and investigate the concept dual to a kernel.

Results on complete families

Theorem 1. *A family $\mathscr{S}$ of graphs is complete if and only if $\mathscr{S} = (\mathscr{S}^{\mathrm{R}})^{\mathrm{C}}$ and $\mathscr{S}$ is closed under homomorphisms.*

Proof. Suppose $\mathscr{S}$ is complete. By Theorem I, $\mathscr{S}$ is closed under homomorphisms. By Theorem K, $\mathscr{S} = (\mathscr{S}^{\mathrm{R}})^{\mathrm{C}}$. Conversely, if $\mathscr{S}$ is closed under homomorphisms, then $\mathscr{S}^{\mathrm{R}}$ is free by Theorem J. Hence $(\mathscr{S}^{\mathrm{R}})^{\mathrm{C}}$ is complete by Theorem K. That is, $\mathscr{S} = (\mathscr{S}^{\mathrm{R}})^{\mathrm{C}}$ is complete. □

Theorem 2. *If $\mathscr{S}$ is a non-empty graph family which is closed under deletion and closed under contraction, then exactly one of the following holds:*
(a) $\mathscr{S}^{\mathrm{o}} = \{K_1\}$;
(b) $\mathscr{S} = \{$*all graphs*$\}$.

Proof. Let $\mathscr{S}$ be a graph family such that $\mathscr{S}^{\mathrm{o}} \neq \{K_1\}$. Since $\mathscr{S}$ is closed under contraction, by Theorem C, $\mathscr{S}^{\mathrm{o}}$ is complete. Then by Theorem H, $\mathscr{S}^{\mathrm{o}}$ contains some spanning supergraphs of complete graphs K_n of every order $n \in \mathbb{N}$. Since $\mathscr{S}$ is non-empty and closed under deletion, $K_1 \in \mathscr{S}$. Hence by Theorem A, $\mathscr{S}^{0} \subseteq \mathscr{S}$. Thus $\mathscr{S}$ contains some spanning supergraphs of the complete graph K_n of every order n. Since $\mathscr{S}$ is closed under deletion, $\mathscr{S} = \{$all graphs$\}$. If (b) holds then $\mathscr{S}^{\mathrm{o}} = \{$all connected graphs$\}$, and so (a) cannot hold simultaneously. □

Examples of graph families closed under both deletion and contraction include {forests} and {planar graphs}.

We say that a *possible edge set* of G is a set of edges of some spanning supergraph of G which are not in $E(G)$.

Lemma 3. *If $\mathscr{C}$ is complete then $\mathscr{C}(t)$ is closed under contraction, for every positive integer t.*

Proof. Let $G \in \mathscr{C}(t)$ and let $e \in E(G)$. By the definition of $\mathscr{C}(t)$, there exists a possible edge set X of G such that $G + X \in \mathscr{C}$ and $|X| \leq t$. Since $\mathscr{C}$ is complete, $\mathscr{C}$ is closed under contraction. Thus $(G + X)/e \in \mathscr{C}$ and so $G/e \in \mathscr{C}(t)$. □

Theorem 4. *If $\mathscr{C}$ is complete then $\mathscr{C} = [\mathscr{C}(t)]^{\circ}$, for every positive integer t.*

Proof. We need to prove $\mathscr{C} \subseteq [\mathscr{C}(t)]^{\circ}$ and $[\mathscr{C}(t)]^{\circ} \subseteq \mathscr{C}$.

Let $H \in \mathscr{C}$. to prove $H \in [\mathscr{C}(t)]^{\circ}$, we must prove that for any supergraph G of H,

$$G \in \mathscr{C}(t) \;\Rightarrow\; G/H \in \mathscr{C}(t).$$

Let G be a supergraph of H. Suppose $G \in \mathscr{C}(t)$. We can find a possible edge set X of G such that $|X| \leq t$ and $G + X \in \mathscr{C}$. By (C2), $(G + X)/H \in \mathscr{C}$. Let

$$X' = \{e \in X\text{: at most one end of } e \text{ is in } V(H)\}.$$

Then X' can be regarded as a possible edge set of G/H and so

$$(G/H) + X' \cong (G + X)/H \in \mathscr{C}.$$

This implies that $G/H \in \mathscr{C}(t)$, since $|X'| \leq |X| \leq t$.

Conversely, suppose that $G/H \in \mathscr{C}(t)$. Then there is a possible edge set X' of G/H with $|X'| \leq t$ such that $(G/H) + X' \in \mathscr{C}$. Note that X' induces (not necessarily uniquely) a possible edge set X of G with $|X'| = |X|$. Since $(G/H) + X' \in \mathscr{C}$,

$$(G + X)/H = (G/H) + X' \in \mathscr{C}. \tag{4}$$

By (4) and $H \in \mathscr{C}$, (C3) implies that $G + X \in \mathscr{C}$, and so $G \in \mathscr{C}(t)$. Thus $H \in [\mathscr{C}(t)]^{\circ}$, by the definition of kernels. Since $H \in \mathscr{C}$ is an arbitrary element, we have

$$\mathscr{C} \subseteq [\mathscr{C}(t)]^{\circ}. \tag{5}$$

Now we prove $[\mathscr{C}(t)]^{\circ} \subseteq \mathscr{C}$.

Let $H \in [\mathscr{C}(t)]^{\circ}$, fix an integer $m \geq 2t + 1$, and let $H_1, H_2, \ldots, H_m$ be m copies of H. Thus we can assume that there are m graph isomorphisms $\phi_i : H \rightarrow H_i$, $(1 \leq i \leq m)$. Pick a vertex v of $V(H)$ and let v_i be $\phi_i(v)$, $(1 \leq i \leq m)$. For $k \in \{1, 2, \ldots, m\}$, define $H^{(k)}$ to be the graph obtained from the union of

$H_1, H_2, \ldots, H_k$ by identifying $v_1, v_2, \ldots, v_k$ into a single vertex which we also call v. By Lemma 3 and Theorem C, $[\mathscr{C}(t)]^{\circ}$ is complete and so by (C3) and $H \in [\mathscr{C}(t)]^0$, $H^{(k)} \in [\mathscr{C}(t)]^{\circ}$, for each k.

By the definition of $\mathscr{C}(t)$, we have $\mathscr{C} \subseteq \mathscr{C}(t)$ and so $K_1 \in \mathscr{C} \subseteq \mathscr{C}(t)$. Hence by Theorem A, $[\mathscr{C}(t)]^{\circ} \subseteq \mathscr{C}(t)$. In particular,

$$H^{(k)} \in \mathscr{C}(t), \quad 1 \leq k \leq m. \tag{6}$$

Thus we can find a possible edge set X, $|X| \leq t$, such that $H^{(m)} + X \in \mathscr{C}$. Since each edge of X is incident with at most two components of $H^{(m)} - v$, since $|X| \leq t$, and since $2|X| \leq 2t < m$, some component ($H_m - v$, say) of $H^{(m)} - v$ is incident with no edges of X. Therefore

$$H \cong H_m \cong (H^{(m)} + X)/H^{(m-1)}. \tag{7}$$

Since $H^{(m)} + X \in \mathscr{C}$ and since $\mathscr{C}$ is closed under contraction,

$$(H^{(m)} + X)/H^{(m-1)} \in \mathscr{C}.$$

Therefore, $H \in \mathscr{C}$, by (7). This means

$$[\mathscr{C}(t)]^{\circ} \subseteq \mathscr{C}. \tag{8}$$

By (5) and (8), the theorem holds. □

Corollary 5. *For the families $\mathscr{S}_{r,t}$, $\Gamma_{r,t}$ and $\mathscr{S}(r, t)$ of Examples 2, 3, and 4, $\mathscr{S}^{\circ}_{r,t} = \mathscr{S}_{r,0}$; $\Gamma^{\circ}_{r,t} = \Gamma_{r,0}$; and $\mathscr{S}^{\circ}(r, t) = \mathscr{S}(r, 0)$.*

Proof. Since $\mathscr{S}_{r,0}$, $\Gamma_{r,0}$ and $\mathscr{S}(r, 0)$ are complete families, this corollary follows from Theorem 4. □

Cokernels

In this section, for $m \in \mathbb{N}$, and for a graph G, mG denotes the vertex-disjoint union of m copies of G.

Let G, H be graphs. By $H \lhd G$ we mean that H is an induced subgraph of G (if $xy \in E(G)$ and $x, y \in V(H)$, then $xy \in E(H)$). A *G-pair* (H', H) is an ordered pair (H', H) of graphs such that $H' \lhd H$ and $G \cong H/H'$.

Definition. The *cokernel* of a family $\mathscr{S}$ is defined as follows:

$$\mathscr{S}_0 = \{K: \forall K\text{-pair } (H', H),\ H' \in \mathscr{S} \Leftrightarrow H \in \mathscr{S}\}.$$

It is trivial that $K_1 \in \mathscr{S}_0$ since for any K_1-pair (G, G), $G \in \mathscr{S} \Leftrightarrow G \in \mathscr{S}$. A graph family may have trivial cokernel. For example, if $\mathscr{S} = \{\text{cycles}\}$, then $\mathscr{S}_0 = \{K_1\}$. If $\mathscr{S} = \{\text{forests}\}$ then $\mathscr{S}_0 = \{\text{forests}\}$ (this is an instance of Theorem 10); if

$$\mathscr{S} = \{\text{graphs with at most } t \text{ cycles}\}$$

where t is fixed, then $\mathscr{S}_0 = \{\text{forests}\}$.

Theorem 6. *Let $\mathscr{S}$ be a family of graphs. Then:*
(a) *$K_1 \in \mathscr{S}$ if and only if $\mathscr{S}_0 \subseteq \mathscr{S}$.*
(b) *$(\mathscr{S}_0)_0 = \mathscr{S}_0$.*

Proof. (a) Suppose $K_1 \in \mathscr{S}$. We shall show that if $H \notin \mathscr{S}$ then $H \notin \mathscr{S}_0$. Note that $H \cong H/K_1$ and so (K_1, H) is an H-pair, and the equivlaence

$$K_1 \in \mathscr{S} \Leftrightarrow H \in \mathscr{S}$$

is false. This implies $H \notin \mathscr{S}_0$ as claimed.

Conversely, suppose $\mathscr{S}_0 \subset \mathscr{S}$. Trivially, $K_1 \in \mathscr{S}_0$ and so $K_1 \in \mathscr{S}$.

(b) We know that $K_1 \in \mathscr{S}_0$. Hence by (a),

$$(\mathscr{S}_0)_0 \subseteq \mathscr{S}_0. \tag{9}$$

To show $\mathscr{S}_0 \subseteq (\mathscr{S}_0)_0$, we choose a graph $K \in \mathscr{S}_0$. Let (G', G) be a K-pair. We want to show

$$G \in \mathscr{S}_0 \Leftrightarrow G' \in \mathscr{S}_0. \tag{10}$$

Suppose $G \in \mathscr{S}_0$. For any G'-pair (H'', H'), since $G' \cong H'/H''$ and $G' \lhd G$, we can find a supergraph H of H'', such that $H'' \lhd H$ and $G \cong H/H''$.

Note that

$$H/H' \cong (H/H'')/(H'/H'') \cong G/G' \cong K \in \mathscr{S}_0. \tag{11}$$

Hence

$$H \in \mathscr{S} \Leftrightarrow H' \in \mathscr{S}. \tag{12}$$

Since $G \in \mathscr{S}_0$, and $G \cong H/H''$,

$$H'' \in \mathscr{S} \Leftrightarrow H \in \mathscr{S}. \tag{13}$$

By (12) and (13),

$$H' \in \mathscr{S} \Leftrightarrow H \in \mathscr{S} \Leftrightarrow H'' \in \mathscr{S}.$$

Thus $G' \in \mathscr{S}_0$.

Conversely, we assume that $G' \in \mathscr{S}_0$. For any G-pair (H'', H), since $G \cong H/H''$, we may assume that

$$E(G) = E(H) - E(H'').$$

Since $G' \lhd G$, $E(G') \subseteq E(G)$.

Let $H' = H[E(G') \cup E(H'')]$. We shall show that $H' \lhd H$. By way of contradiction, suppose that there is an edge $uv \in E(H) - E(H')$, where $u, v \in V(H')$. If $u, v \in V(H'')$, then by $H'' \lhd H$, we have $uv \in E(H'') \subseteq E(H')$. Hence not both u, v are in $V(H'')$. Thus in $G' \cong H'/H''$, u and v are distinct vertices in $V(G')$. Since $G' \lhd G$, $uv \in E(G')$, which implies $uv \in E(H')$, a contradiction.

Since $H'' \lhd H$ and $H'' \subseteq H'$, it follows that $H'' \lhd H'$. Hence

$$H'' \lhd H' \lhd H \quad \text{and} \quad G' \cong H'/H''.$$

Since $G' \in \mathscr{S}_0$,

$$H'' \in \mathscr{S} \Leftrightarrow H' \in \mathscr{S}. \tag{14}$$

Note that (11) and (12) are still valid. Hence by (12) and (14),

$$H \in \mathscr{S} \Leftrightarrow H' \in \mathscr{S} \Leftrightarrow H'' \in \mathscr{S}.$$

It follows by the definition that $G \in \mathscr{S}_0$, and so (10) follows.

Since (G', G) is an arbitrary K-pair, by definition, $K \in (\mathscr{S}_0)_0$ and so $(\mathscr{S}_0) \subseteq (\mathscr{S}_0)_0$.

The proof of Theorem 6 is complete. □

Remark 1. Theorem 6 is the dual form of Theorem A.

Definition. Let G and H be two vertex-disjoint graphs. Let H be connected and $v \in V(G)$. We denote by $G_{v,H}$ a graph obtained from G by replacing v by H such that each edge in $E(G)$ incident with v has exactly one end in $V(H)$. A 1-*sum* of G and H is a graph obtained from G and H by identifying one vertex of G and one vertex of H.

The outcome of taking 1-sums may not be unique.

Theorem 7. *If $\mathscr{F}$ is a free family, then*:

(a) $\mathscr{F} = \mathscr{F}_0$.

(b) *Let G and H be graphs, let $v \in V(G)$, and suppose that H is connected. If $G, H \in \mathscr{F}$ then $G_{v,H} \in \mathscr{F}$.*

(c) *If $\mathscr{F} \neq \{K_1\}$ and if H_1, H_2 are two connected graphs in $\mathscr{F}$, then the vertex-disjoint union of H_1 and H_2 is also in $\mathscr{F}$.*

(d) *$\mathscr{F}$ is closed under taking* 1-*sums.*

Proof. (a) Let $\mathscr{F}$ be a free family. Then $K_1 \in \mathscr{F}$. By Theorem 6,

$$\mathscr{F}_0 \subseteq \mathscr{F}. \tag{15}$$

Conversely, let $G \in \mathscr{F}$. To show that $G \in \mathscr{F}_0$, we need to show that for any G-pair, (H', H),

$$H \in \mathscr{F} \Leftrightarrow H' \in \mathscr{F}. \tag{16}$$

Fix a G-pair (H', H). Then (F2) implies '$\Rightarrow$' of (16) and that $H' \in \mathscr{F}$ and (F3) together imply '$\Leftarrow$' of (16). Hence $\mathscr{F} \subseteq \mathscr{F}_0$ and (a) follows.

(b) Suppose G and H satisfy the hypothesis of (b). Then $G_{v,H}/H \cong G$. By (F3), since $\mathscr{F}$ is free, and since $G, H \in \mathscr{F}$, this implies $G_{v,H} \in \mathscr{F}$.

(c) Since $\mathscr{F} \neq \{K_1\}$, (F2) implies $2K_1 \in \mathscr{F}$. Since H_1 is in $\mathscr{F}$ and is connected, by (F3), the vertex-disjoint union of K_1 and H_1 is in $\mathscr{F}$. Since H_2 is connected and is in $\mathscr{F}$, by (F3) again, the vertex-disjoint union of H_1 and H_2 is in $\mathscr{F}$.

(d) Let $H_1, H_2 \in \mathscr{F}$ be two vertex-disjoint graphs with $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$, and let G denote the 1-sum of H_1 and H_2 obtained by identifying v_1 and v_2. Since $H_2 \in \mathscr{F}$ and since $G/H_2 = H_1 \in \mathscr{F}$, it follows from (F3) that G is in $\mathscr{F}$. $\square$

Corollary 8. *Let $\mathscr{F}$ be a free family. Then one of the following holds.*

(a) $\mathscr{F} = \{iK_1 : 1 \leq i \leq m\}$, *for some* $m \in \mathbb{N}$;

(b) $\mathscr{F} = \{iK_1 : i \in \mathbb{N}\}$;

(c) *$\mathscr{F}$ contains the family of all forests.*

Proof. It is routine to check that both (a) and (b) define free families. If no member of $\mathscr{F}$ contains an edge, then clearly (a) and (b) holds. Suppose that there is a graph $G \in \mathscr{F}$ with $E(G) \neq \emptyset$. Then by (F2), $2K_1$, $K_2 \in \mathscr{F}$. By (d) of Theorem 7, $\mathscr{F}$ contains all trees. By (c) of Theorem 7, $\mathscr{F}$ contains all forests. $\square$

Remark 2. Theorem 7 is the dual form of Theorem B and Corollary 8 is the dual form of Theorem H.

Theorem 9. *For any graph family $\mathscr{S}$, if $\mathscr{S}$ or $\mathscr{S}_0$ is closed under deletion, then $\mathscr{S}_0$ is free.*

Proof. We assume that $\mathscr{S}$ is closed under deletion first and shall verify that the three axioms for free families will be satisfied. Since $K_1 \in \mathscr{S}_0$, (F1) is satisfied.

If $\mathscr{S}$ is closed under deletion then let $G \in \mathscr{S}_0$ and let $G' \subseteq G$. For any G'-pair (H'', H'), we can find a supergraph H of H' such that

$$H'' \lhd H' \lhd H \quad \text{and} \quad G \cong H/H''.$$

Since $G \in \mathscr{S}_0$,

$$H'' \in \mathscr{S} \Leftrightarrow H \in \mathscr{S}. \tag{17}$$

Since $\mathscr{S}$ is closed under deletion and by (17),

$$H'' \in \mathscr{S} \Rightarrow H \in \mathscr{S} \Rightarrow H' \in \mathscr{S}.$$

By the assumption that $\mathscr{S}$ is closed under deletion,

$$H' \in \mathscr{S} \Rightarrow H'' \in \mathscr{S}.$$

Hence $G' \in \mathscr{S}_0$ and so (F2) holds in either case.

For (F3), we let $G' \lhd G$, $G' \in \mathscr{S}_0$ and $G/G' \in \mathscr{S}_0$. By Theorem 6, $\mathscr{S}_0 = (\mathscr{S}_0)_0$. Since $G/G' \in \mathscr{S}_0 = (\mathscr{S}_0)_0$ and since $G' \in \mathscr{S}_0$, we have $G \in \mathscr{S}_0$ and so (F3) holds.

Now we assume that $\mathscr{S}_0$ is closed under deletion. By the result we have just done, $(\mathscr{S}_0)_0$ is free. By Theorem 6, $\mathscr{S}_0 = (\mathscr{S}_0)_0$ and so $\mathscr{S}_0$ is free. $\square$

Remark 3. Theorem 9 is the dual form of Theorem C.

Example: Let

$$\mathcal{P} = \{\text{planar graphs}\}.$$

Then

$$\mathcal{P}_0 = \{\text{forests}\}.$$

Proof. First we show that $\{\text{forests}\} \subseteq \mathcal{P}_0$.

Since $\mathcal{P}$ is closed under deletion, it follows from Theorem 9 that $\mathcal{P}_0$ is free. To show $\{\text{forests}\} \subseteq \mathcal{P}_0$, it suffices by Corollary 8 to show $K_2 \in \mathcal{P}_0$.

Suppose $G = K_2$. For any G-pair (H', H), $H \in \mathcal{P} \Rightarrow H' \in \mathcal{P}$.

On the other hand, suppose $H' \in \mathcal{P}$. Since $G = K_2$, for each G-pair (H', H), H is of the form that the two components of H' are joined by a cut edge of H. Hence $H' \in \mathcal{P} \Rightarrow H \in \mathcal{P}$. Thus $G \in \mathcal{P}_0$. By Corollary 8, $\{\text{forests}\} \subseteq \mathcal{P}_0$.

Now we show that $\mathcal{P}_0 \subseteq \{\text{forests}\}$. It suffices to prove that the graphs in $\mathcal{P}_0$ are acyclic. By way of contradiction, suppose that a graph G containing a cycle C is in $\mathcal{P}_0$. There is a graph, say $G(v, w)$, containing a pair of distinct vertices v and w, such that G is obtained from $G(v, w)$ when v and w are identified, and such that the edges of C induce a (v, w)-path in $G(v, w)$. Let H' be the graph obtained from $K_{3,3}$ by removing an edge, say $v'w'$. Construct H from $G(v, w) \cup H'$ by setting $v = v'$ and $w = w'$. Then $H' \lhd H$ and $H/H' \cong G$. Since H contains a subdivision of $K_{3,3}$, $H \notin \mathcal{P}$. But $H' \in \mathcal{P}$, and so $G \notin \mathcal{P}_0$. Therefore, $\mathcal{P}_0 \subseteq \{\text{forests}\}$.

Summing up, we have $\mathcal{P}_0 = \{\text{forests}\}$. □

Theorem 10. *Let $\mathcal{F}$ be a family of graphs closed under deletion. These are equivalent*:

(a) $\mathcal{F}$ is the cokernel of some graph family closed under deletion;

(b) *$\mathcal{F}$ is a free family*;

(c) $\mathcal{F} = \mathcal{F}_0$.

Proof. We shall prove that (a)$\Leftrightarrow$(c) and (b)$\Leftrightarrow$(c).

(a)$\Rightarrow$(c) Suppose that $\mathcal{F}$ is the cokernel of some graphs family $\mathcal{S}$ closed under deletion. Then $\mathcal{F} = \mathcal{S}_0$. Thus by Theorem 6(b),

$$\mathcal{F}_0 = (\mathcal{S}_0)_0 = \mathcal{S}_0 = \mathcal{F}.$$

(c)$\Rightarrow$(a) Since $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}$ is a cokernel of itself. Since $\mathcal{F}$ is closed under deletion, (a) holds.

(b)$\Rightarrow$(c) This is Theorem 7(a).

(c)$\Rightarrow$(b) Since $\mathcal{F}$ is closed under deletion, both (F1) and (F2) hold. Suppose that H is an induced subgraph of G. Also suppose that $H \in \mathcal{F}$ and $G/H \in \mathcal{F}$. Since $\mathcal{F} = \mathcal{F}_0$, for the G/H-pair (H, G), $H \in \mathcal{F} \Rightarrow G \in \mathcal{F}$. Hence (F3) holds. □

Remark 4. Theorem 10 is the dual form of Theorem D.

Definition. If $k < |E(G)|$, we define

$$G/k = \{G/G[X]: X \subseteq E(G) \quad \text{and} \quad |X| \leq k\}.$$

and if $k \geq |E(G)|$, we define

$$G/k = \{\omega(G)K_1\},$$

where $\omega(G)$ is the number of components of G.

Now let $\mathscr{F}$ be a free family, and let

$$\mathscr{F}(k) = \{H: \exists G \in \mathscr{F} \text{ with } H \in G/k\}.$$

Lemma 11. *If $\mathscr{F}$ is free then $\mathscr{F}(k)$ is closed under deletion.*

Proof. Let $H' \subseteq H$ and $H \in \mathscr{F}(k)$. Then there is $G \in \mathscr{F}$, such that $H = G/X$ for some $X \subseteq E(G)$ and $|X| \leq k$. It follows that there exist $G' \subseteq G$ and $X' \subseteq X$ with

$$H' = G'/X.$$

Since $\mathscr{F}$ is closed under deletion, $G' \in \mathscr{F}$ and so $H' \in G'/k$. Therefore $H' \in \mathscr{F}(k)$. □

Theorem 12. *If $\mathscr{F}$ is free then $\mathscr{F}(k)_0 = \mathscr{F}$.*

Proof. We first prove $\mathscr{F} \subseteq \mathscr{F}(k)$. Fix $G \in \mathscr{F}$ and let (H', H) be a G-pair. Suppose $H \in \mathscr{F}(k)$. Then by Lemma 11 above, $H' \in \mathscr{F}(k)$.

Conversely, assume $H' \in \mathscr{F}(k)$. Then there are $K' \in \mathscr{F}$ and $X \subseteq E(K')$ with $|X| \leq k$ and $H' = K'/X$. It implies that there is a supergraph K of K' with $K' \lhd K$ and $K/X \cong H$. Hence

$$G \cong H/H' \cong K/K'.$$

Since $G \in \mathscr{F}$ and $K' \in \mathscr{F}$, and since $\mathscr{F}_0 = \mathscr{F}$, we have $K \in \mathscr{F}$ and so $H = K/X \in \mathscr{F}(k)$.

Thus $G \in \mathscr{F}(k)$ and so

$$\mathscr{F} \subseteq \mathscr{F}(k).$$

Then we shall show that $\mathscr{F}(k)_0 \subseteq \mathscr{F}$.

Let $G \in \mathscr{F}(k)_0$. Since $\mathscr{F}(k)_0$ is free, $\mathscr{F}(k)_0$ is closed under edge-disjoint unions. Hence we have $(k+1)G \in \mathscr{F}(k)_0$. Since $K_1 \in \mathscr{F}(k)$, ($\mathscr{F}(k)$ is closed under deletion), by Theorem 6(a), $\mathscr{F}(k)_0 \subseteq \mathscr{F}(k)$. Hence

$$(k+1)G \in \mathscr{F}(k).$$

It follows that there exist $G'' \in \mathscr{F}$ and $X \subseteq E(G'')$ with $|X| \leq k$ such that $(k+1)G = G''/X$. Since $|X| \leq k$, at least one copy of G is subgraph of G''. It implies that $G \in \mathscr{F}$. □

Remark 5. Lemmas 3, and 11, Theorems 4 and 12 are dual to each other, respectively.

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